An instability criterion for a finite amplitude localized disturbance in a shear flow of electrically conducting fluids

by

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Abstract

The stability of shear flows of electrically conducting fluids, with respect to finite amplitude three-dimensional localized disturbances is considered. The time evolution of the fluid impulse integral, characterizing such disturbances, for the case of low magnetic Reynolds number is obtained by integrating analytically the vorticity equation. Analysis of the resulted equation reveals a new instability criterion.

1 Introduction

The use of magnetohydrodynamics (MHD) for transition and turbulent control is quite attractive from the point of view of applications, in particular when new technologies may allow direct turbulent control in sea water. Most of the work in this field has been concerned with flows of electrically conducting fluids subjected to transverse magnetic fields. This configuration is used in MHD generators, accelerators and pumps. Recently, Nosenchuck and Brown (1993), demonstrated experimentally that the application of wall-normal Lorentz force prohibit lift-up and bursting of near wall fluid, which are characteristics of the end-stage of transition and near wall turbulence. However, in such a configuration, the magnetic effects are due mainly to the coupling between the mean velocity profile and the magnetic field, rather than the damping of turbulence.

On the other hand, when the mean flow and the magnetic field are aligned, the direct effect of the magnetic field is on the disturbed velocity field. Fraim and Heiser (1968) studied experimentally the effect of a strong longitudinal magnetic field on the flow of mercury in a circular tube. They found that the magnetic field can significantly increase the critical Reynolds number (Re_{cr}) for transition. More examples showing similar results are summarized in the book of Branover (1978), who concluded that linear theories of stability of MHD flows yield values of Re_{cr} which are much

higher than the measured values. This suggests that a proper explanation of the above mentioned experimental results must take nonlinear effects into account.

The evolution of a finite amplitude three-dimensional localized vortex disturbance embedded in an external incompressible shear flow was considered by Levinski & Cohen (1995, hereinafter reffered to as LC). Using the fluid impulse as an integral characteristic of such a disturbance, they found that parallel shear flows are always unstable with respect to localized disturbances, the typical dimension of which δ, is much smaller than a dimensional length scale Δ, corresponding to an $\mathcal{O}(1)$ change of the external velocity. Moreover, their analysis predicts that the growing vortex disturbance is inclined at 45^0 to the external flow direction, in a plane normal to the transverse axis. It was also shown that although viscosity plays a crucial role in the generation of the initial localized disturbance and in determining the mean flow field, it plays no role in the time evolution of its fluid impulse integral. In other words, once the mean field is established, the subsequent evolution of the disturbance fluid impulse integral is largely an inviscid one. These predictions agree with previous experimental observations concerning the growth of hairpin vortices in laminar and turbulent boundary layers, see e.g. Head & Bandyopadhyay (1981), Acarlar & Smith (1987) and Hagen & Kurosaka (1993). The application of this approach to Taylor-Couette flow revealed a new instability criterion, which was recently verified experimentally by Cohen *et al.* (1996).

The purpose of the work reported here is to examine the effect of an externally imposed magnetic field on the onset of such disturbances. The analysis is restricted to incompressible shear flows characterized by a low magnetic Reynolds number, $Re_m = \mu\sigma\Delta U \ll 1$, where μ is the magnetic permeability, σ is the electrical conductivity and U is a characteristic velocity scale of the external flow. The magnetic Reynolds number can be considered as the ratio between the diffusion time scale of the magnetic field $\mu\sigma\Delta^2$, and the hydrodynamic time scale Δ/U. For small values of Re_m, the magnetic field induced by a disturbance diffuses rapidly. This leads to the dissipation of the disturbance kinetic energy and consequently may stabilize the flow.

2 Analysis

For $Re_m \ll 1$ and stationary external magnetic field, it was shown by Braginskii (1960) that the electromagnetic force per unit volume is $\mathbf{f} = \sigma\left[(\mathbf{U}_T \times \mathbf{B}) \times \mathbf{B} - \nabla\Phi_T \times \mathbf{B}\right]$, where $\mathbf{B}$ is the magnetic induction of the external field and $\mathbf{U}_T$ is the total velocity vector as defined below. The scalar potential Φ_T is determined from the condition that the charge is neutralized, which for a

uniform liquid yields

$$\nabla^2 \Phi_T = \mathbf{B} \cdot \mathbf{\Omega}_T, \tag{1}$$

where $\mathbf{\Omega}_T = \nabla \times \mathbf{U}_T$ is the total vorticity vector. For this case, the three-dimensional vorticity equation for an incompressible flow is given by

$$\frac{\partial \mathbf{\Omega}_T}{\partial t} + (\mathbf{U}_T \cdot \nabla)\mathbf{\Omega}_T - (\mathbf{\Omega}_T \cdot \nabla)\mathbf{U}_T - \frac{\sigma}{\rho}(\mathbf{B} \cdot \nabla)\left(\mathbf{U}_T \times \mathbf{B} - \nabla \Phi_T\right) = \nu \Delta \mathbf{\Omega}_T, \tag{2}$$

where ρ and ν are the density and kinematic viscosity of the fluid, respectively.

We consider the flow field as being the sum of two contributions: the external shear field in which $\mathbf{\Omega} = \nabla \times \mathbf{U}$, and a finite amplitude disturbed field in which $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Thus, the total velocity and vorticity vectors can be written as $\mathbf{U}_T = \mathbf{U} + \mathbf{u}$ and $\mathbf{\Omega}_{\boldsymbol{T}} = \mathbf{\Omega} + \boldsymbol{\omega}$, where the undisturbed external flow field is assumed to be a known solution of (1) and (2) when $\mathbf{u} = 0$. The initial disturbed vorticity $\boldsymbol{\omega}(\mathbf{x}, t_0) = \boldsymbol{\omega}_0(\mathbf{x})$ is assumed to be confined to the small region of order $\delta \ll \Delta$ as well as $\delta \ll \Delta_1$, where Δ_1 is a typical length scale corresponding to an $\mathcal{O}(1)$ change of the external magnetic field.

Owing to the smallness of the disturbed region, the external velocity, vorticity and magnetic fields are approximated by Taylor series expansions. Following LC, we use a Galilean frame, moving with the disturbance, i.e., $\mathbf{U}(0) = 0$, and consider the initially embedded vorticity region as surrounded by an infinite field having a constant velocity shear and a constant magnetic induction. Consequently,

$$U_i(\mathbf{x}) = \sum_{j=1}^{3} \frac{\partial U_i(0)}{\partial x_j} x_j, \quad \Omega_i(\mathbf{x}) = \Omega_i(0), \quad B_i(\mathbf{x}) = B_i(0) \quad \textit{where} \quad i = 1, 2, 3. \tag{3}$$

When the equations for the undisturbed external flow and magnetic field are subtracted from (1) and (2) respectively, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{U} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{U} - (\mathbf{\Omega} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \frac{\sigma}{\rho}(\mathbf{B} \cdot \nabla)\left(\mathbf{u} \times \mathbf{B} - \nabla \phi\right) = 0, \tag{4}$$

and

$$\nabla^2 \phi = \mathbf{B} \cdot \omega. \tag{5}$$

In (4) the viscous term is omitted since, as was shown by LC, the corresponding contribution of

the viscous term to the dynamics of the fluid impulse integral vanishes in view of the asymptotic behavior of the disturbance vorticity far from the origin.

We shall follow the evolution in time of the fluid impulse integral of the disturbance, defined as

$$\frac{d\mathbf{P}}{dt} = \frac{1}{2}\int \mathbf{x} \times \frac{\partial \boldsymbol{\omega}(\mathbf{x}, t)}{\partial t} dV, \tag{6}$$

where $\mathbf{x}$ is the position vector, dV is a volume element and the time derivative of $\boldsymbol{\omega}(\mathbf{x}, t)$ is determined from (4).

Since the time evolution of the fluid impulse is an integral over the whole volume, we must first verify that most of the contribution to this integral comes from the localized disturbed region. Indeed, all of the vorticity contributing to this integral, except for the part generated via the fourth and the seventh (electromagnetic) terms in (4), is confined to the disturbed region.

In order to estimate the contributions of the fourth and the seventh terms in (4), we examine their far-field behavior. The expression for the scalar potential ϕ is obtained from the general solution of (5)

$$\phi = \frac{1}{4\pi}\int \frac{\mathbf{B} \cdot \omega(\mathbf{x}_1)}{|\mathbf{x} - \mathbf{x}_1|^3} dV_1 = \mathbf{B} \cdot \mathbf{M}, \tag{7}$$

where the asymptotic series of $\mathbf{M}$, expressed in terms of the fluid impulse, is given (Batchelor, 1967) by

$$\mathbf{M} = \frac{1}{4\pi}\mathbf{P} \times \frac{\mathbf{x}}{|\mathbf{x}|^3} + \mathcal{O}(\frac{1}{|\mathbf{x}|^3}). \tag{8}$$

Similarly, the far-field velocity induced by the localized vortex disturbance is

$$\mathbf{u}(\mathbf{x}) \sim -\frac{1}{4\pi}\left[\frac{\mathbf{P}}{|\mathbf{x}|^3} - \frac{3(\mathbf{P} \cdot \mathbf{x})\mathbf{x}}{|\mathbf{x}|^5}\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^4}\right). \tag{9}$$

Substitution of (8) and (9) into (4) shows that the far-field vorticity diminishes in magnitude as $|\mathbf{x}|^{-4}$. Consequently, the integral (6) is not absolutely convergent and depends on the way in which the volume of integration is allowed to tend to infinity.

In order to overcome this difficulty we follow the procedure proposed in LC. Accordingly, we subdivide the velocity and vorticity fields into two parts, $\boldsymbol{\omega} = \boldsymbol{\omega}^I + \boldsymbol{\omega}^{II}$ and $\mathbf{u} = \mathbf{u}^I + \mathbf{u}^{II}$, so that $\boldsymbol{\omega}^{I,II} = \nabla \times \mathbf{u}^{I,II}$. Therefore, for each part we require that

$$\nabla \cdot \boldsymbol{\omega}^I = \nabla \cdot \boldsymbol{\omega}^{II} = 0. \tag{10}$$

The first part, indicated by the superscript I, is associated with the concentrated vorticity confined within and in the vicinity of the initially disturbed region, whereas the second part, indicated by the superscript II, is associated with the far-field vorticity generated by the problematic terms (the fourth and seventh) in (4). Accordingly, we set the initial distribution of the vorticity fields as:

$$\boldsymbol{\omega}^I(\mathbf{x}, t = t_0) = \boldsymbol{\omega}_0(\mathbf{x}) \qquad and \qquad \boldsymbol{\omega}^{II}(\mathbf{x}, t = t_0) = 0, \tag{11}$$

and follow the evolution of $\boldsymbol{\omega}^I(\mathbf{x}, t)$. In addition, we subdivide the whole space into two regions, inside and outside a spherical domain of radius $R \geq \delta$, enclosing the disturbance. In the outer region the corresponding system of the vorticity equations is given by

$$\frac{\partial \boldsymbol{\omega}^I}{\partial t} + (\mathbf{U}\cdot\nabla)\boldsymbol{\omega}^I - (\boldsymbol{\omega}^I\cdot\nabla)\mathbf{U} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - (\boldsymbol{\Omega}\cdot\nabla)(\mathbf{u}^I - \mathbf{u}_0)$$

$$- \frac{\sigma}{\rho}(\mathbf{B}\cdot\nabla)\left[(\mathbf{u}^I - \mathbf{u}_0)\times\mathbf{B} - \nabla\left(\mathbf{B}\cdot(\mathbf{M}^I - \mathbf{M}_0)\right)\right] = 0, \tag{12}$$

$$\frac{\partial \boldsymbol{\omega}^{II}}{\partial t} + (\mathbf{U}\cdot\nabla)\boldsymbol{\omega}^{II} - (\boldsymbol{\omega}^{II}\cdot\nabla)\mathbf{U} - (\boldsymbol{\Omega}\cdot\nabla)(\mathbf{u}^{II} + \mathbf{u}_0)$$

$$- \frac{\sigma}{\rho}(\mathbf{B}\cdot\nabla)\left[(\mathbf{u}^{II} + \mathbf{u}_0)\times\mathbf{B} - \nabla\left(\mathbf{B}\cdot(\mathbf{M}^{II} + \mathbf{M}_0)\right)\right] = 0, \tag{13}$$

where $\mathbf{u}_0$ and $\mathbf{M}_0$ are the leading terms in the asymptotic series of (8) and (9),

$$\mathbf{u}_0 = -\frac{1}{4\pi}\left[\frac{\mathbf{p}}{|\mathbf{x}|^3} - \frac{3(\mathbf{p}\cdot\mathbf{x})\mathbf{x}}{|\mathbf{x}|^5}\right] \quad ; \quad \mathbf{M}_0 = \frac{1}{4\pi}\mathbf{p}\times\frac{\mathbf{x}}{|\mathbf{x}|^3}, \tag{14}$$

and the fluid impulse $\mathbf{p}$ corresponds only to $\boldsymbol{\omega}^I$, i.e.,

$$\mathbf{p} = \frac{1}{2}\int \mathbf{x}\times\boldsymbol{\omega}^I(\mathbf{x})dV. \tag{15}$$

Since $\mathbf{u}_0$ and $\mathbf{M}_0$ cancel the leading terms of the far-field vorticity, generated via the problematic terms in (12), the asymptotic behavior of $\boldsymbol{\omega}^I$ is given by

$$\left|\boldsymbol{\omega}^I(\mathbf{x}, t)\right| \sim \mathcal{O}\left(\frac{1}{|\mathbf{x}|^5}\right) \quad for \quad |\mathbf{x}| \gg \delta, \tag{16}$$

and therefore the fluid impulse integral (15) is absolutely convergent.

For the inner region we write

$$\frac{\partial \boldsymbol{\omega}^I}{\partial t} + (\mathbf{U}\cdot\nabla)\boldsymbol{\omega}^I - (\boldsymbol{\omega}^I\cdot\nabla)\mathbf{U} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u}^I$$

$$-\frac{\sigma}{\rho}(\mathbf{B}\cdot\nabla)\left(\mathbf{u}^I\times\mathbf{B} - \nabla(\mathbf{B}\cdot\mathbf{M}^I)\right) + \nabla\Psi = 0, \tag{17}$$

$$\frac{\partial \boldsymbol{\omega}^{II}}{\partial t} + (\mathbf{U}\cdot\nabla)\boldsymbol{\omega}^{II} - (\boldsymbol{\omega}^{II}\cdot\nabla)\mathbf{U} - (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u}^{II} - \frac{\sigma}{\rho}(\mathbf{B}\cdot\nabla)\left(\mathbf{u}^{II}\times\mathbf{B} - \nabla(\mathbf{B}\cdot\mathbf{M}^{II})\right) - \nabla\Psi = 0, \tag{18}$$

so that the sum of the two equations in each region yields Eq. (2) for that region, and together with the initial conditions given in (11), yields the original problem for the entire space.

For the outer region, condition (10) is always satisfied. For this condition to be satisfied in the inner region $\nabla^2\Psi$ must be equal to zero, as can be shown by applying the operator $(\nabla\cdot)$ to (17) and (18). Then, Ψ is determined by solving the Neumann problem for which the normal derivative of Ψ at $|\mathbf{x}| = R$ is matched with the scaler product of the unit vector normal to the boundary surface $\mathbf{n}$, and the terms in (12), containing $\mathbf{u}_0$ and $\mathbf{M}_0$, i.e.

$$\mathbf{n}\cdot\left.\frac{\partial\Psi}{\partial\mathbf{n}}\right|_{|\mathbf{x}|=R} = \mathbf{n}\cdot\left.\left[(\boldsymbol{\Omega}\cdot\nabla)\mathbf{u}_0 + \frac{\sigma}{\rho}(\mathbf{B}\cdot\nabla)\left(\mathbf{u}_0\times\mathbf{B} - \nabla(\mathbf{B}\cdot\mathbf{M}_0)\right)\right]\right|_{|\mathbf{x}|=R}. \tag{19}$$

Accordingly, using (14) the expression for Ψ is given by

$$\Psi = \frac{3}{8\pi R^4}\left[(\boldsymbol{\Omega}\cdot\mathbf{p})|\mathbf{x}|^2 - 3(\boldsymbol{\Omega}\cdot\mathbf{x})(\mathbf{p}\cdot\mathbf{x}) + \frac{4\sigma}{\rho}(\mathbf{B}\cdot\mathbf{x})(\mathbf{x}\cdot(\mathbf{p}\times\mathbf{B}))\right]. \tag{20}$$

Since the integral (15) is absolutely convergent, it is convenient to use an infinite sphere as the volume of integration. Consequently, the time evolution of $\mathbf{p}$ is given by

$$\frac{d\mathbf{p}}{dt} = \frac{1}{2}\lim_{R_1\to\infty}\int_{|\mathbf{x}|\le R_1}\mathbf{x}\times\frac{\partial\boldsymbol{\omega}^I(\mathbf{x},t)}{\partial t}dV. \tag{21}$$

Substitution of (12) and (17) into (21) yields

$$\frac{d\mathbf{p}}{dt} = -\frac{1}{2}\lim_{R_1\to\infty}\int_{|\mathbf{x}|\le R_1}\mathbf{x}\times\left[(\mathbf{U}\cdot\nabla)\boldsymbol{\omega}^I - (\boldsymbol{\omega}^I\cdot\nabla)\mathbf{U} - (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u^I} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\right]dV$$

$$+\frac{\sigma}{2\rho}\lim_{R_1\to\infty}\int_{|\mathbf{x}|\le R_1}\mathbf{x}\times\left[(\mathbf{B}\cdot\nabla)\left(\mathbf{u}^I\times\mathbf{B} - \nabla(\mathbf{B}\cdot\mathbf{M}^I)\right)\right]dV - \frac{1}{2}\lim_{R_1\to\infty}\int_{R\le|\mathbf{x}|\le R_1}\mathbf{x}\times[(\boldsymbol{\Omega}\cdot\nabla)\mathbf{u_0}$$

$$+\frac{\sigma}{\rho}(\mathbf{B}\cdot\nabla)\mathbf{u}_0\times\mathbf{B}\Big]\,dV+\frac{\sigma}{2\rho}\lim_{R_1\to\infty}\int_{R\le|\mathbf{x}|\le R_1}\mathbf{x}\times(\mathbf{B}\cdot\nabla)\nabla(\mathbf{B}\cdot\mathbf{M}_0)\,dV-\frac{1}{2}\int_{|\mathbf{x}|\le R}\mathbf{x}\times\nabla\Psi\,dV. \quad (22)$$

Each one of the integrals in (22) is evaluated for a finite value of R_1 and its limit as $R_1 \to \infty$ is then taken. The result of the first integral in (22) was already obtained in LC. As shown in the Appendix for such domains of integration, the integral contribution of the last three integrals in (22) is identically zero. Therefore, the artificial vorticity field has no direct impact on the evolution of the fluid impulse associated with the concentrated vorticity $\boldsymbol{\omega}^I(\mathbf{x},t)$. Moreover, as was shown in LC, the influence of the vorticity field $\boldsymbol{\omega}^{II}(\mathbf{x},t)$ on the evolution of $\boldsymbol{\omega}^I(\mathbf{x},t)$ can be neglected. The second integral in (22) is calculated using a similar procedure to that described in the Appendix and in LC. Finally, (22) becomes

$$\frac{d\mathbf{p}}{dt}=-\frac{1}{2}\nabla(\mathbf{p}\cdot\mathbf{U})-\frac{1}{2}(\mathbf{p}\cdot\nabla)\mathbf{U}-\frac{2\sigma B^2}{5\rho}\mathbf{p}+\frac{\sigma}{5\rho}(\mathbf{B}\cdot\mathbf{p})\mathbf{B}. \quad (23)$$

3 Application to a representative example

In the following we consider a simple example in which (23) is applied to a parallel shear flow of electrically conducting fluids and a new instability criterion for finite amplitude localized disturbances is found. For a parallel plane shear flow, the external velocity field is given by $\mathbf{U}=(U(y),0,0)$, for which a right-handed coordinate system is used with $\mathbf{x}=(x,y,z)$, where the vector entries are the downstream, cross-flow and spanwise directions, respectively. The direction of the magnetic induction vector is chosen to be parallel to the downstream direction, i.e. $\mathbf{B}=(B,0,0)$. As was mentioned above, such a flow does not exhibit a direct coupling between the mean flow and the magnetic field. Consequently, the direct effect of the magnetic field on the turbulent structure can be revealed. In this case, equation (23) for the fluid impulse vector $\mathbf{p}=(p_x,p_y,p_z)$, is reduced to

$$\frac{dp_x}{dt}=-\frac{1}{2}p_y\frac{dU}{dy}-\frac{\sigma B^2}{5\rho}p_x,\qquad \frac{dp_y}{dt}=-\frac{1}{2}p_x\frac{dU}{dy}-\frac{2\sigma B^2}{5\rho}p_y,\qquad \frac{dp_z}{dt}=-\frac{2\sigma B^2}{5\rho}p_z, \quad (24)$$

for which the eigenvalues $\{\lambda_i\}_{i=1}^3$ can be found from the characteristic equation

$$\left(\lambda_i+\frac{2\sigma B^2}{5\rho}\right)\left[\lambda_i^2+\frac{3\sigma B^2}{5\rho}\lambda_i+2\left(\frac{\sigma B^2}{5\rho}\right)^2-\left(\frac{1}{2}\frac{dU}{dy}\right)^2\right]=0. \quad (25)$$

Hence, the flow under investigation is stable with respect to three-dimensional localized disturbances only if the real part of λ_i is not positive. Therefore, for stability we require that

$$2\left(\frac{\sigma B^2}{5\rho}\right)^2 \geq \left(\frac{1}{2}\frac{dU}{dy}\right)^2 \quad or \quad N = \frac{\sigma B^2}{\rho dU/dy} \geq \frac{5}{2\sqrt{2}} \; , \tag{26}$$

where N is a dimensionless interaction parameter which represents the ratio between the electromagnetic and the inertia forces.

The term 'stability' (and 'instability') here means that the fluid impulse of a closed localized (in all three directions) vortex disturbance will not grow (or grow) in time. The fluid impulse is a very suitable characteristic of localized vortex structures such as hairpin vortices in boundary layers, since it combines the geometrical dimensions of the structure together with the magnitude of its vorticity field. As such, this stability definition cannot describe 'wavy' disturbances or 'quasi' two-dimensional structures for which the fluid impulse integral is not defined. It should be noted however, that the above stability definition is not equivalent to the conventional criteria of linear stability and energy stability. In fact, the growth of the fluid impulse does not necessarily guaranty growth in energy of the localized disturbance. For example, viscous diffusion leads to the decay of the localized disturbance energy while its fluid impulse remains the same.

As an example of the new stability criterion, the value of the interaction parameter required for stability of Poiseuille flow in a tube is $\sigma B^2 D/\rho\bar{U} \geq 14.1$, where D is the tube diameter and $\bar{U}$ is the mean velocity. For flow of mercury in a circular tube subjected to a strong longitudinal magnetic field, Fraim & Heiser reported an increase of the critical Reynolds number for transition from 2250 to 10350 when the magnetic induction B was increased from zero to $1.75 weber/m^2$, and the corresponding interaction parameter at the upper limit was 9. As can be seen in Fig. 10 of their article, this value of the interaction parameter is an order of magnitude larger than the value predicted by Stuart's linear theory (Stuart, 1954). A direct comparison of these results with our prediction is questionable because of at least two reasons. The first is that the experimental mean velocity profiles were not reported by Fraim & Heiser. The second is that the criterion presented here is applicable only with respect to localized disturbances whereas transitional flows include various types of disturbances. Nevertheless, the predicted value of the interaction parameter found in this paper is of the same order of magnitude as that of the experimental ones for large Reynolds numbers.

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Appendix

In this Appendix we show that the last three integrals of (22), defined as $\mathbf{J}^1$, $\mathbf{J}^2$ and $\mathbf{J}^3$, respectively, are identically zero. In Cartesian tensor notation, the sum of $\mathbf{J}^2$ and $\mathbf{J}^3$ is

$$J_i^2 + J_i^3 = \frac{\sigma}{2\rho}\epsilon_{ijk}\int_{R\le|\mathbf{x}|\le R_1} x_j \frac{\partial}{\partial x_k}(\mathbf{B}\cdot\nabla)(\mathbf{B}\cdot\mathbf{M}_0)\,dV - \frac{1}{2}\epsilon_{ijk}\int_{|\mathbf{x}|\le R} x_j\frac{\partial\Psi}{\partial x_k}\,dV\,, \tag{A1}$$

where ϵ_{ijk} is the alternating tensor and the usual summation convention is applied.

Integration by parts and using Gauss' divergence theorem yields

$$J_i^2 + J_i^3 = \frac{\sigma}{2\rho}\epsilon_{ijk}\left[\oint_{|\mathbf{x}|=R_1} n_k x_j(\mathbf{B}\cdot\nabla)(\mathbf{B}\cdot\mathbf{M}_0)dS - \oint_{|\mathbf{x}|=R} n_k x_j(\mathbf{B}\cdot\nabla)(\mathbf{B}\cdot\mathbf{M}_0)dS\right.$$

$$\left. -\delta_{jk}\int_{R\le|\mathbf{x}|\le R_1}(\mathbf{B}\cdot\nabla)(\mathbf{B}\cdot\mathbf{M}_0)\,dV\right] - \frac{1}{2}\epsilon_{ijk}\left[\oint_{|\mathbf{x}|=R} n_k x_j\Psi dS - \delta_{jk}\int_{|\mathbf{x}|\le R}\Psi\,dV\right], \tag{A2}$$

where δ_{ij} is the Kronecker delta function. Using the properties of the alternating and symmetrical tensors $\epsilon_{ijk}\delta_{jk}\equiv 0$, and $\epsilon_{ijk}n_k n_j \equiv 0$, it follows that $\mathbf{J}^2 = \mathbf{J}^3 = 0$.

A similar procedure for $\mathbf{J}^1$ gives

$$\mathbf{J}^1 = \frac{1}{2}\left(\oint_{|\mathbf{x}|=R_1} - \oint_{|\mathbf{x}|=R}\right)[d\mathbf{S}(\boldsymbol{\Omega}\cdot\mathbf{M}_0) - (d\mathbf{S}\cdot\boldsymbol{\Omega})(\mathbf{M}_0 + \mathbf{x}\times\mathbf{u}_0)$$

$$+\frac{\sigma}{\rho}\left[B^2 d\mathbf{S}\times\mathbf{M}_0 - \mathbf{B}((d\mathbf{S}\times\mathbf{M}_0)\cdot\mathbf{B}) - (d\mathbf{S}\cdot\mathbf{B})(\mathbf{x}\times(\mathbf{u}_0\times\mathbf{B}))\right]\Bigg]. \tag{A3}$$

When (14) is substituted into (A3), the result of the two surface integrals become independent of the surface radii and consequently cancel each other, i.e., $\mathbf{J}^1 = 0$.

References

ACARLAR, M.S. & SMITH, C.R. 1987 A study of hairpin vortices in a laminar boundary layer. Part 1. Hairpin vortices generated by a hemisphere protuberance. *J. Fluid Mech.* **175**, 1.

BATCHELOR, G.K. 1967 Introduction to fluid dynamics. PP. 517-520. *Cambridge, University Press.*

BRAGINSKI, S.I. 1960 Magnetohydrodynamics of weakly conducting liquids. *Soviet Phys. JETP* **37** (10), No. 5, 1005.

BRANOVER, H. 1978 Magnetohydrodynamic flow in ducts. *John Wiley & Sons, New York.*

COHEN, J., LEVINSKI, V., MALKIEL, E. & NAIMARK, E. The evolution of hairpin vortices in rotating flows. *Proceedings of the 36th Israel Ann. Conf. on Aerospace Sci.*, 35.

FRAIM, F.W. & HEISER, W.H. 1968 The effect of a strong longitudinal magnetic field on the flow of mercury in a circular tube. *J. Fluid Mech.* **33**, 397.

HAGEN, J.P. & KUROSAKA, M. 1993 Corewise cross-flow transport in hairpin vortices - The "tornado effect". *Phys. Fluids A* **5** (12), 3167.

HEAD, M.R & BANDYOPADHYAY, P. 1981 New aspect of turbulent boundary-layers structure. *J. Fluid Mech.* **107**, 297.

LEVINSKI, V. & COHEN, J. 1995 The evolution of a localized vortex disturbance in external shear flows. Part 1. Theoretical considerations and preliminary experimental results. *J. Fluid Mech.* **289**, 159.

NOSENCHUCK, D & BROWN, G. 1993 Active control of transition using the Lorentz force. *Proceedings of the Syracuse University Minnowbrook Workshop on end-stage boundary layer transition*, Blue Mountain Lake, New York.

STUART, J.T. 1954 On the stability of viscous flow between parallel planes in the presence of a co-planar magnetic field. *Proc. Roy. Soc. Lond. Ser. A* **221**, 189.